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Gauss-Seidel Iteration Method

Consider a system of n linear algebraic equation in n unknown

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\;\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \rightarrow (1)$$

Where a_{ij} $(i, j = 1(1)n)$ are the known coefficients, b_i, $(i = 1(1)n)$ are the known values and x_i $(i = 1(1)n)$ are the unknowns to be determined.

For "Gauss-Seidel Iteration method", we assume the quantities a_{ii} in (1) are pivot elements. The equation (1) may be written as —

$$\left.\begin{aligned} x_1 &= -\frac{1}{a_{11}}(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n - b_1) \\ x_2 &= -\frac{1}{a_{22}}(a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n - b_2) \\ x_n &= -\frac{1}{a_{nn}}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn-1}x_{n-1} - b_n) \end{aligned}\right\} \rightarrow (2)$$

Next starting this iteration method with initial approximation $x_1 = x_1^{(0)}$, $x_2 = x_2^{(0)}$ $\cdots$ $x_n = x_n^{(0)}$

If we use this initial value of $x_i^{(0)}$ $i = 1(1)n$ in (2) we get 1st approximation which are

$$x_1^{(1)} = -\frac{1}{a_{11}}\left(a_{12}x_2^{(0)} + a_{13}x_3^{(0)} + \cdots + a_{1n}x_n^{(0)} - b_1\right)$$

$$x_2^{(1)} = -\frac{1}{a_{22}}\left(a_{21}x_1^{(1)} + a_{23}x_3^{(0)} + \cdots + a_{2n}x_n^{(0)} - b_2\right)$$

$$x_n^{(1)} = -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \cdots + a_{nn-1}x_{n-1}^{(1)} - b_n\right)$$

(Note: In calculation of $x_2^{(1)}, x \cdots x_n^{(1)}$ we use the new value of $x_1, x_2, \cdots x_n$, which is basic difference between this method and Gauss-Jacobi method).

In this ways we get (K+1)th approximation as :-

$$x_1^{(k+1)} = -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \cdots + a_{1n}x_n^{(k)} + b_1\right)$$

$$x_2^{(k+1)} = -\frac{1}{a_{22}}\left(a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)} - b_2\right)$$

$$\vdots$$

$$x_n^{(k+1)} = -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \cdots + a_{n\,n-1}x_{n-1}^{(k+1)} - b_n\right)$$

Since we replace the vector $x^{(k)}$ in the right side of (1) element by element, this method is also called the method of successive displacement.

In matrix notation, we first express the coefficient matrix A in the form

$A = D + L + U$ where D is a diagonal part of A, L is strictly lower triangular part of A, and U is strictly upper triangular part of A,

[Note: It is important to keep in mind that the matrices L and U used here are in no way related to the LU decomposition of the coefficient matrix.]

Let $\{x^{(k+1)}\}$ and $\{x^{(k)}\}$ are the seq^n of root

and $\lim_{k\to\infty} x^{(k+1)} = \lim_{k\to\infty} x^{k} = X$

Let's suppose $A = M - N$ where $M = D + L$ and $N = -U$

Then $AX = b$

$\Rightarrow (M-N)X = b$

$\Rightarrow MX = NX + b$

$\Rightarrow X = M^{-1}NX + M^{-1}b$

$\Rightarrow X^{(k+1)} = -(D+L)^{-1}UX^{(k)} + (D+L)^{-1}b$

$= HX^{(k)} + C$

Where $H = -(D+L)^{-1}U$
$c = (D+L)^{-1}b.$

Alternatively be written as

$$x^{(k+1)} = -(D+L)^{-1}U\,x^{(k)} + Ix^{(k)} - Ix^{(k)} + (D+L)^{-1}b.$$

$$= x^{(k)} - [I + (D+L)^{-1}U]x^{(k)} + (D+L)^{-1}b.$$

$$= x^{(k)} - [(D+L)^{-1}(D+L) + (D+L)^{-1}U]x^{(k)} + (D+L)^{-1}b$$

$$= x^{(k)} - [(D+L)^{-1}\{D+L+U\}]x^{(k)} + (D+L)^{-1}b.$$

$$= x^{(k)} - (D+L)^{-1}(D+L+U)x^{(k)} + (D+L)^{-1}b.$$

$$= x^{(k)} - (D+L)^{-1}Ax^{(k)} + (D+L)^{-1}b.$$

$$= x^{(k)} + (D+L)^{-1}(b - Ax^{(k)})$$

$$\therefore x^{(k+1)} - x^{(k)} = (D+L)^{-1}(b - Ax^{(k)}).$$

$$\to v^{(k)} = (D+L)^{-1}r^{(k)}$$

where $v^{(k)} = x^{(k+1)} - x^{(k)}$ and
$r^{(k)} = b - Ax^{(k)}$ is the residual vector.

$\therefore (D+L)v^{(k)} = r^{(k)}.$

and solve for $v^{(k)}$ by forward substitution

$\therefore$ The solution is then found from
$x^{(k+1)} = x^{(k)} + v^{(k)}$

These equation describe the Gauss-Seidel method in an error format.

Ex: Find the Soln of the System of eqn by Gauss-Seidel method —

$$20x_1 + x_2 - 2x_3 = 17$$
$$3x_1 + 20x_2 - x_3 = 18$$
$$2x_1 + 3x_2 + 20x_3 = 25$$

Soln: Take the initial approximation as $x^{(0)} = 0$

i.e. $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$.

Now rewrite the ~~system as the form~~ iteration formula

$$x_1^{(k+1)} = -\frac{1}{20}\left(x_2^{(k)} - 2x_3^{(k)} - 17\right).$$
$$x_2^{(k+1)} = -\frac{1}{20}\left(3x_1^{(k+1)} - x_3^{(k)} + 18\right).$$
$$x_3^{(k+1)} = -\frac{1}{20}\left(2x_1^{(k+1)} - 3x_2^{(k+1)} - 25\right).$$

1st Iterat

$$x_1^{(1)} = -\frac{1}{20}\left(x_2^{(0)} - 2x_3^{(0)} - 17\right) = -\frac{1}{20}(-17) = \underline{0.8}$$
$$x_2^{(1)} = -\frac{1}{20}\left(3(0.8) - 0 + 18\right) = \underline{-1.0275}$$
$$x_3^{(1)} = -\frac{1}{20}\left(2\cdot x_1^{(1)} - 3x_2^{(1)} - 25\right) = -\frac{1}{20}\{2\cdot(0.8) - 3\cdot(-1.0275) - 25\} = \underline{1.0109}$$

2nd Iteratio

$$x_1^{(2)} = -\frac{1}{20}\left(-1.0275 - 2\cdot(1.0109) - 17\right) = \underline{1.0025}$$
$$x_2^{(2)} = -\frac{1}{20}\left(3\cdot(1.0025) - (1.0109) + 18\right) = \underline{-0.9998}$$
$$x_3^{(2)} = -\frac{1}{20}\left(2\cdot(1.0025) - 3\cdot(-0.9998) - 25\right) = 0.9998$$

3rd

$$x_1^{(3)} = -\frac{1}{20}\left(0.9998 - 2\cdot(0.9998) + 17\right) = 1.000.$$
$$x_2^{(3)} = -\frac{1}{20}\left(3\cdot 1 - 0.9998 + 18\right) = -1.000$$
$$x_3^{(3)} = -\frac{1}{20}\left(2\cdot 1 - 3\cdot 1 - 25\right) = 1.000$$

Solution is $x_1 = 1$, $x_2 = -1$, $x_3 = 1$

$$x^{(k+1)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} x^{(k)} + \begin{bmatrix} \frac{7}{2} \\ \frac{9}{4} \\ \frac{13}{8} \end{bmatrix}$$

$$x^{(k+1)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} x^{(0)} + \begin{pmatrix} \frac{7}{2} \\ \frac{9}{4} \\ \frac{13}{8} \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{pmatrix} \frac{7}{2} \\ \frac{9}{4} \\ \frac{13}{8} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{2} \\ \frac{9}{4} \\ \frac{13}{8} \end{pmatrix}$$

$$x^{(2)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} \begin{pmatrix} \frac{7}{2} \\ \frac{9}{4} \\ \frac{13}{8} \end{pmatrix} + \begin{pmatrix} \frac{7}{2} \\ \frac{9}{4} \\ \frac{13}{8} \end{pmatrix}$$

$$= \begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix}.$$

$$x^{(3)} = \begin{bmatrix} 5.3125 \\ 4.3125 \\ 2.6563 \end{bmatrix}.$$

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